Exercise Session for Financial Data Analysis

Summer term 2011

Problem Set 5

Write to haas@stat.uni-muenchen.de if you want to present. You can also indicate multiple exercises (ordered according to your preference) in case your most preferred problem has already been assigned.

Problem 1 A specific simple version of a smooth transition GARCH (STGARCH)[1] model is given by

$$\begin{aligned}
\epsilon_t &= \sigma_t \eta_t, \quad \eta_t \overset{iid}{\sim} \mathrm{N}(0,1) && (1)\\
\sigma_t^2 &= \omega + \alpha_1 \epsilon_{t-1}^2 + \alpha_2 \epsilon_{t-1}^2 G(\epsilon_{t-1};\gamma) + \beta \sigma_{t-1}^2 && (2)\\
G(\epsilon_{t-1};\gamma) &= \frac{1}{1+e^{\gamma \epsilon_{t-1}}}, \quad \gamma > 0, && (3)\\
\omega &> 0, \quad \alpha_1, \beta \geq 0, \quad \alpha_1 + \alpha_2 \geq 0. && (4)
\end{aligned}$$

(a) Explain the effect of the *transition function* $G(\epsilon_{t-1};\gamma)$ in (1)–(3). Which model is obtained for $\gamma \to \infty$?

Problem 2 Describe the main problems that arise when the GARCH model is generalized to capture the covariance dynamics of multivariate time series.

Problem 3 Suppose that the *Single–Index Model* (SIM) describes the percentage returns of individual stocks, r_i, $i = 1, \ldots, N$, that is, with r_M being the return of the market,

$$\begin{aligned}
r_i &= \alpha_i + \beta_i r_M + \epsilon_i, \quad \mathrm{E}(\epsilon_i) = 0, \quad \mathrm{Var}(\epsilon_i) = \sigma_i^2, \quad i = 1, \ldots, N;\\
\mathrm{Cov}(\epsilon_i, \epsilon_j) &= 0, \quad i \neq j; \quad \mathrm{Cov}(\epsilon_i, r_M) = 0, \quad i = 1, \ldots, N;\\
\mathrm{E}(r_M) &= \mu_M, \quad \mathrm{Var}(r_M) = \sigma_M^2.
\end{aligned}$$

(a) Find $\mathrm{Var}(r_i)$, $i = 1, \ldots, N$, and $\mathrm{Corr}(r_i, r_j)$, the correlation between r_i and r_j.

(b) Show that, if β_i and β_j have the same sign, the correlation between r_i and r_j is increasing in the variance of the market return, σ_M^2.

[1] G. Gonzales–Rivera (1998): Smooth Transition GARCH Models, *Studies in Nonlinear Dynamics & Econometrics* 3, 161–178.

Problem 4

Your portfolio consists of two risky assets, the percentage returns[2] of which are bivariate normally distributed with mean vector, $\boldsymbol{\mu}$, and covariance matrix, $\boldsymbol{\Sigma}$, given by

$$\boldsymbol{\mu} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad \boldsymbol{\Sigma} = \begin{bmatrix} 2 & 1 \\ 1 & 5 \end{bmatrix},$$

respectively. The weight vector of your portfolio is $\boldsymbol{w} = [0.5, 0.5]'$, and you have invested 5000 \$ in this portfolio. Compute the 1% Value–at–Risk for this position.

Problem 5 Consider the following asymmetric ARCH(1) process:

$$\begin{aligned} \epsilon_t &= \eta_t \sigma_t, \quad \eta_t \overset{iid}{\sim} \mathrm{N}(0,1) && (5) \\ \sigma_t^2 &= \omega + \alpha(\epsilon_{t-1} - \theta)^2 && (6) \\ \omega &> 0, \quad 0 < \alpha < 1/\sqrt{3}, \quad \theta \in \mathbb{R}. && (7) \end{aligned}$$

(i) For the model defined by (5)–(7), find $\mathrm{Cov}(\sigma_t^2, \epsilon_{t-1})$, i.e., the covariance between σ_t^2 and ϵ_{t-1}, and interpret your result.

(ii) Find $\mathrm{Cov}(\sigma_t^2, \epsilon_{t-\tau})$ for $\tau \geq 1$.

Problem 6 Consider a two–component scale[3] normal mixture distribution with density

$$f(y) = \frac{\lambda}{\sqrt{2\pi}\sigma_1} \exp\left\{-\frac{(y-\mu)^2}{2\sigma_1^2}\right\} + \frac{1-\lambda}{\sqrt{2\pi}\sigma_2} \exp\left\{-\frac{(y-\mu)^2}{2\sigma_2^2}\right\}, \qquad (8)$$

where $\lambda \in (0,1)$, and $\sigma_1^2 \neq \sigma_2^2$.

(a) Show that a random variable described by (8) implies a kurtosis which is larger than that of a normal random variable. (Without loss of generality, you may assume that $\mu = 0$.)

[2]That is, return $r_t = 100 \times (P_t - P_{t-1})/P_{t-1}$, where P_t is the asset price at time t.

[3]This is a scale mixture since only the scale parameter σ^2 is component–specific, whereas the mean μ is the same in each component.

Problem 7 Find the error in the following argument:

"The weekly return of our portfolio is generated by the GARCH(1,1) model

$$
\begin{aligned}
r_t &= \mu + \eta_t \sigma_t, \quad \eta_t \overset{iid}{\sim} \text{Normal}(0,1) && (9)\\
\sigma_t^2 &= \omega + \alpha \epsilon_{t-1}^2 + \beta \sigma_{t-1}^2. && (10)
\end{aligned}
$$

In order to calculate the two–week–ahead Value–at–Risk at time t, we need

$$
\begin{aligned}
\tilde{\sigma}_t^2(2) &:= \text{Var}_t(r_{t+1} + r_{t+2}) = \text{Var}_t(r_{t+1}) + \text{Var}_t(r_{t+2}) && (11)\\
&= \text{E}_t(\sigma_{t+1}^2) + \text{E}_t(\sigma_{t+2}^2) = \sigma_{t+1}^2 + \text{E}_t(\sigma_{t+2}^2), && (12)
\end{aligned}
$$

where $\text{E}_t(\sigma_{t+2}^2) = \omega + (\alpha + \beta)\sigma_{t+1}^2$. Thus, the conditional distribution of $r_{t:t+2} := r_{t+1} + r_{t+2}$ is

$$
r_{t:t+2} | I_t \sim \text{Normal}(2\mu, \tilde{\sigma}_t^2(2)), \tag{13}
$$

where $I_t = \{\epsilon_s : s \leq t\}$ is the infoprmation available up to time t. Hence, the ξ–quantile of the two–week return distribution, q_ξ, is

$$
q_\xi = 2\mu + z_\xi \tilde{\sigma}_t^2(2), \tag{14}
$$

where z_ξ is the corresponding quantile of the standard normal distribution."