Intermediate Econometrics
Summer term 2011

The vector of regression residuals

$$\begin{aligned}
\widehat{u} &= y - X\widehat{\beta} = y - X(X'X)^{-1}X'y = (I - X(X'X)^{-1}X')y && (1)\\
&= (I - X(X'X)^{-1}X')(X\beta + u) = (I - X(X'X)^{-1}X')u && (2)\\
&= M_X u, && (3)
\end{aligned}$$

where

$$M_X = I - X(X'X)^{-1}X' \qquad (4)$$

It is straightforward to check that matrix M_X defined in (4) is symmetric and *idempotent*, i.e., $M_X' = M_X$ and

$$M_X = M_X^2 = M_X^3 = \cdots \qquad (5)$$

Now consider the first–order conditions for OLS,

$$X'X\widehat{\beta} = X'y. \qquad (6)$$

Partitioning the regressor matrix X in a set of variables X_1 and X_2, i.e., $X = [X_1, X_2]$, with a corresponding partition of the vector of OLS coefficients, $\widehat{\beta} = [\widehat{\beta}_1', \widehat{\beta}_2']'$, we have

$$X'X = \begin{bmatrix} X_1' \\ X_2' \end{bmatrix} \begin{bmatrix} X_1 & X_2 \end{bmatrix} = \begin{bmatrix} X_1'X_1 & X_1'X_2 \\ X_2'X_1 & X_2'X_2 \end{bmatrix}, \qquad (7)$$

and so (9) becomes

$$\begin{bmatrix} X_1'X_1 & X_1'X_2 \\ X_2'X_1 & X_2'X_2 \end{bmatrix} \begin{bmatrix} \widehat{\beta}_1 \\ \widehat{\beta}_2 \end{bmatrix} = \begin{bmatrix} X_1' \\ X_2' \end{bmatrix} y, \qquad (8)$$

or

$$\begin{aligned}
X_1'X_1\widehat{\beta}_1 + X_1'X_2\widehat{\beta}_2 &= X_1'y && (9)\\
X_2'X_1\widehat{\beta}_1 + X_2'X_2\widehat{\beta}_2 &= X_2'y. && (10)
\end{aligned}$$

The first equation of (9) gives

$$\widehat{\beta}_1 = (X_1'X_1)^{-1}X_1'(y - X_2\widehat{\beta}_2), \qquad (11)$$

and plugging this expression for $\widehat{\beta}_1$ into the second equation of (9) gives

$$\begin{aligned}
X_2'(I - X_1(X_1'X_1)^{-1}X_1')X_2\widehat{\beta}_2 &= X_2'(I - X_1(X_1'X_1)^{-1}X_1')y \quad (12)\\
\widehat{\beta}_2 &= (X_2'M_{X_1}X_2)^{-1}X_2'M_{X_1}y \quad (13)\\
\widehat{\beta}_2 &= (X_2'M_{X_1}'M_{X_1}X_2)^{-1}X_2'M_{X_1}'M_{X_1}y, \quad (14)
\end{aligned}$$

where $M_{X_1} = I - X_1'(X_1'X_1)^{-1}X_1$ is analogous to (4) with equivalent interpretation: Any vector multiplied from the right by M_{X_1} produces the residuals of a regression of this vector on the variables in X_1. Therefore, we have the following interpretation of (14): The columns of $M_{X_1}X_2$ and $M_{X_1}y$ are the residuals of a regression of the variables in X_2 and y, respectively, on the variables in X_1. That is, we obtain the OLS estimators for the variables in X_2, i.e., $\widehat{\beta}_2$, by first removing the linear impact of X_1 from both X_2 and y. The residuals of these regressions are uncorrelated with X_1. Hence, for calculating the vector $\widehat{\beta}_2$, only that part of the variation of X_2 and y is used which is uncorrelated with all the other independent variables in X. This interpretation holds for any set of independent variables and therefore for each single variable; that is, each $\widehat{\beta}_j$ measures the effect of x_j on y after all the other variables x_ℓ, $\ell \neq j$, have been partialled out.

Unbiased estimation of the residual variance: From (1) and the symmetry and idempotency of M_X,

$$\widehat{u}'\widehat{u} = u'M_X'M_Xu = u'M_Xu = \mathrm{tr}(u'M_Xu) = \mathrm{tr}(M_Xuu'), \quad (15)$$

using the permutation rule

$$\mathrm{tr}(ABC) = \mathrm{tr}(CAB) = \mathrm{tr}(BCA). \quad (16)$$

Hence, as $E(uu') = \sigma^2 I$,

$$\mathrm{E}(\mathrm{tr}(M_Xuu')) = \mathrm{tr}(M_X\mathrm{E}(uu')) = \sigma^2\mathrm{tr}(M_X), \quad (17)$$

where $M_X = I_n - X(X'X)^{-1}X'$, and so, using (16) once again,

$$\begin{aligned}
\mathrm{tr}(I_n - X(X'X)^{-1}X') &= \mathrm{tr}(I_n) - \mathrm{tr}(X(X'X)^{-1}X')\\
&= n - \mathrm{tr}(X(X'X)^{-1}X')\\
&= n - \mathrm{tr}((X'X)^{-1}X'X)\\
&= n - \mathrm{tr}(I_{k+1}) = n - k - 1.
\end{aligned}$$

Summarizing, $\mathrm{E}(\widehat{u}'\widehat{u}) = (n-k-1)\sigma^2$, and therefore $\mathrm{E}(\widehat{\sigma}^2) = \sigma^2$, with

$$\widehat{\sigma}^2 = \frac{1}{n-k-1}\sum_{i=1}^{n}\widehat{u}_i^2 = \frac{\widehat{u}'\widehat{u}}{n-k-1}. \quad (18)$$