The Probit Model

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Course outline

1. Notation and statistical foundations
2. Introduction to the Probit model
3. Application
4. Coefficients and marginal effects
5. Goodness-of-fit
6. Hypothesis tests

Notation and statistical foundations

1. $y_i = \beta_1 + \beta_2 x_{i2} + \ldots + \beta_k x_{ik} + \varepsilon_i$ Gujarati

$y_t = \beta_0 + \beta_1 x_{t1} + \ldots + \beta_k x_{tk} + u_t$ Wooldridge

2. Matrix

$$Y = X\beta + \varepsilon$$

$$Y = x'\beta + \varepsilon$$

$$Y = X\hat{\beta} + \hat{u}$$

$$y_i = x_i'\beta + \varepsilon_i$$

$$\begin{array}{cc} x_i'\beta & x_i'\beta \\ \begin{pmatrix} 1 & x_1 & x_2 \end{pmatrix}\begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{pmatrix} & \begin{pmatrix} 1 & x_2 & x_3 \end{pmatrix}\begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} \end{array}$$

Notation and statistical foundations – Vectors

- Column vector:

$$\underset{nx1}{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

- Transposed (row vector): $\underset{1xn}{a}' = \begin{bmatrix} a_1 & a_2 & \dots & a_n \end{bmatrix}$

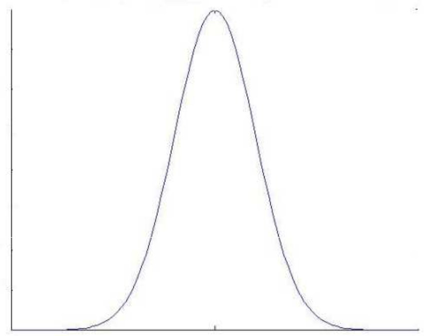
- Inner product:

$$a'b = \begin{bmatrix} a_1 & a_2 & \dots & a_n \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \sum a_i b_i$$

Notation and statistical foundations – density function

- PDF: probability density function f(x)
- Example: Normal distribution:

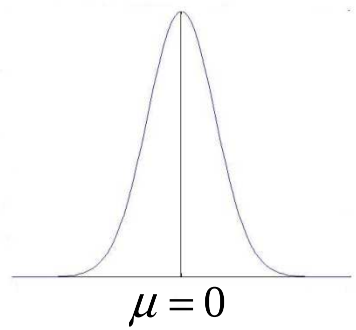
$$\phi(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left[\frac{(x-\mu)^2}{\sigma^2}\right]}$$



- Example: Standard normal distribution:

N(0,1), $\mu = 0$, $\sigma = 1$

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$



Notation and statistical foundations – distibutions

- Standard logistic distribution:

$$f(x) = \frac{e^x}{\left(1+e^x\right)^2}, \mu = 0, \sigma^2 = \frac{\pi^2}{3}$$

Exponential distribution:

$$f(x) = \begin{cases} \frac{1}{\theta} e^{-\frac{x}{\theta}}, x \geq 0 \\ 0, x \leq 0 \end{cases}, \theta > 0, \mu = \theta, \sigma^2 = \theta^2$$

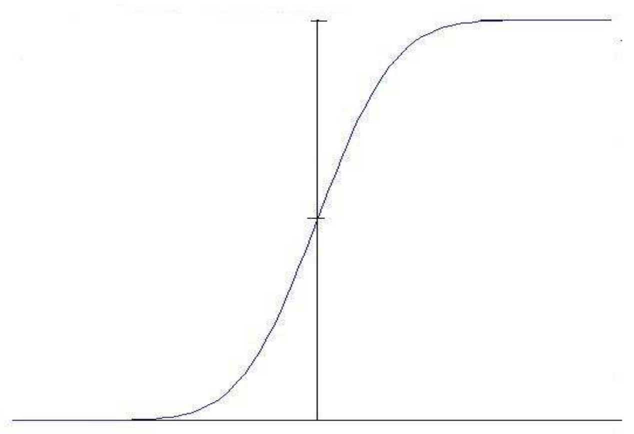
- Poisson distribution:

$$f(x) = \frac{e^{-\theta}\theta^x}{x!}, \mu = \theta, \sigma^2 = \theta$$

Notation and statistical foundations – CDF

- CDF: cumulative distribution function F(x)
- Example: Standard normal distribution:

$$\Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$



- The cdf is the integral of the pdf.

Notation and statistical foundations – logarithms

- Rule I:

$$y = x\,z$$

$$\log y = \log x + \log z$$

- Rule II:

$$y = x^{n}$$

$$\log y = n \log x$$

- Rule III:

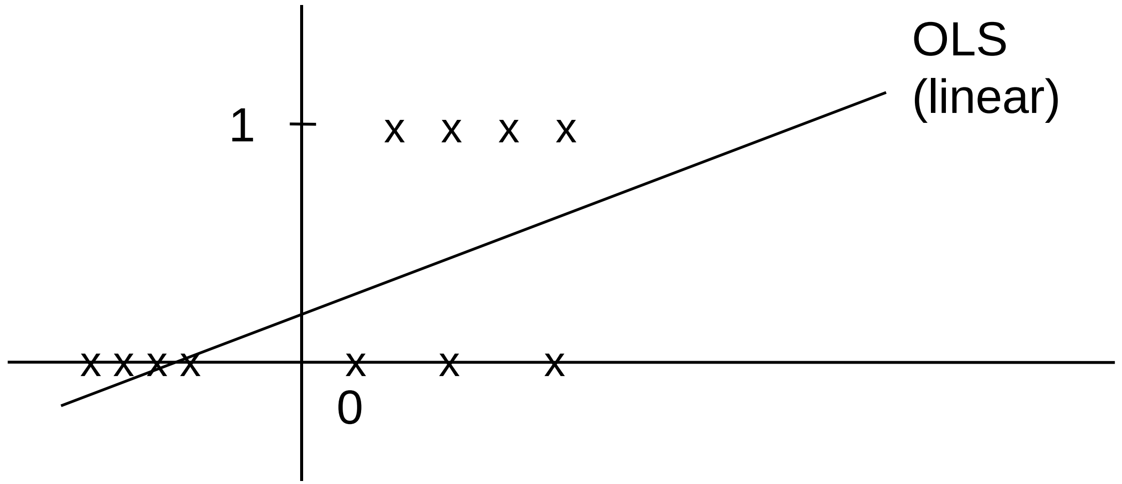
$$y = a\,x^{b}$$

$$\log y = \log a + b \log x$$

Introduction to the Probit model – binary variables

- Why not use OLS instead?

$$y = \begin{cases} 1 \\ 0 \end{cases}$$



- Nonlinear estimation, for example by maximum likelihood.

Introduction to the Probit model – latent variables

- Latent variable: Unobservable variable y* which can take all values in (-∞, +∞).
- Example: y* = Utility(Labour income) - Utility(Non labour income)
- Underlying latent model:

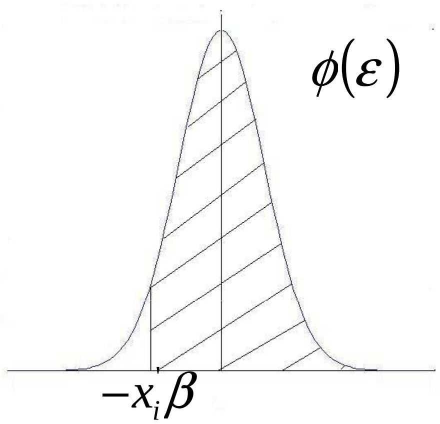
$$y_i = \begin{cases} 1, & y_i^* > 0 \\ 0, & y_i^* \leq 0 \end{cases}$$

$$y_i^* = x_i^{'}\beta + \varepsilon_i$$

Introduction to the Probit model – latent variables

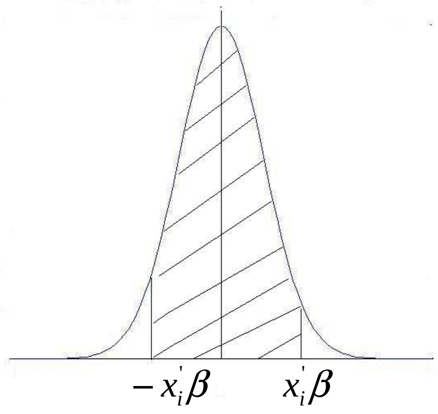
➤ Probit is based on a latent model:

$$P(y_i = 1 \mid x) = P(y_i^* > 0 \mid x)$$
$$= P(x_i'\beta + \varepsilon_i > 0 \mid x)$$
$$= P(\varepsilon_i > -x_i'\beta \mid x)$$
$$= 1 - F(-x_i'\beta)$$



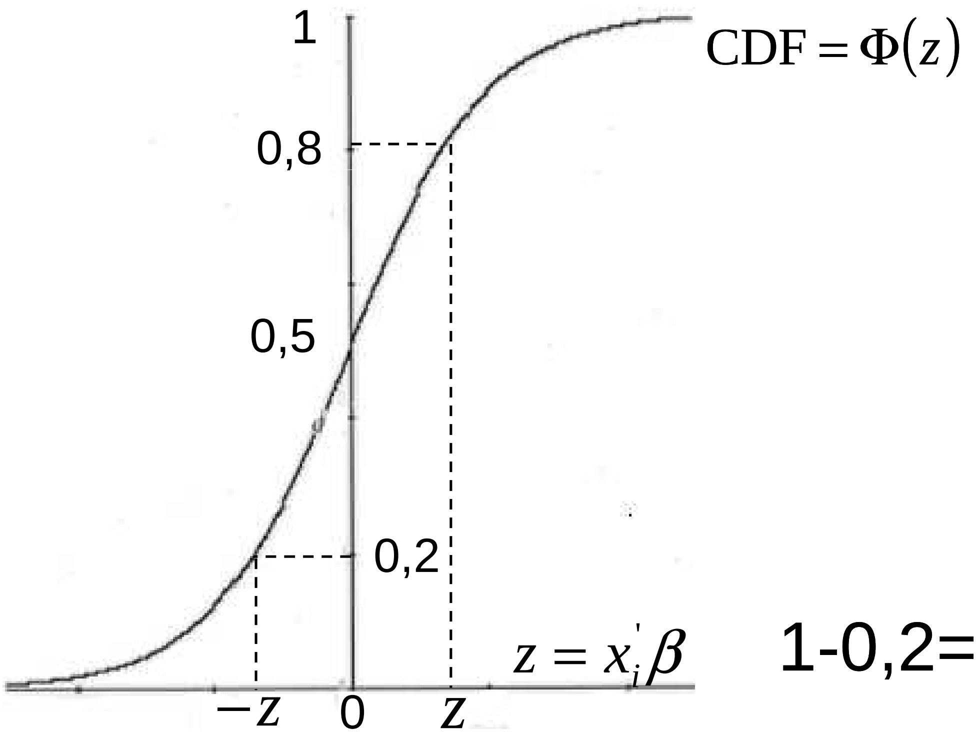
Assumption: Error terms are independent and normally distributed:

$$P(y_i = 1 \mid x) = 1 - \Phi(-\frac{x_i'\beta}{\sigma}), \sigma \equiv 1$$
$$= \Phi(x_i'\beta) \quad \text{because of symmetry}$$



Introduction to the Probit model – CDF

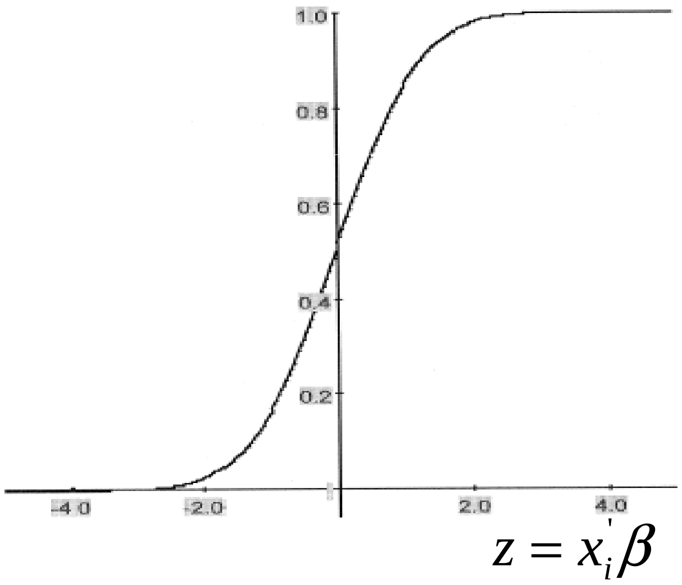
➢ Example:

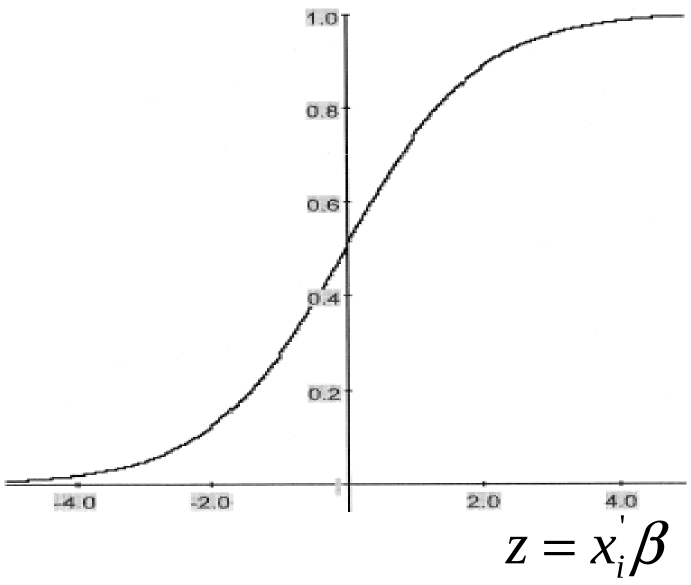


1-0,2=0,8

Introduction to the Probit model – CDF Probit vs. Logit

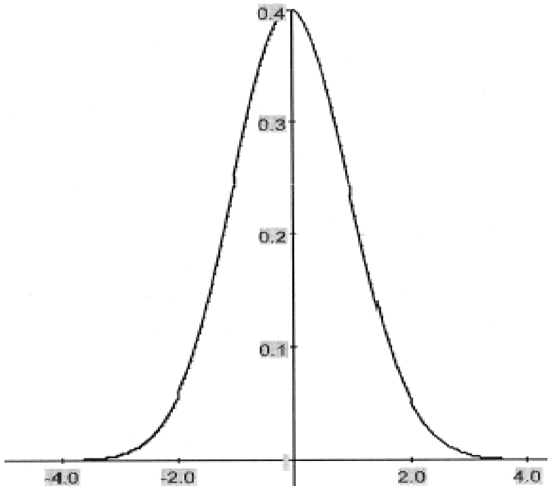
- F(z) lies between zero and one
- CDF of Probit: CDF of Logit:



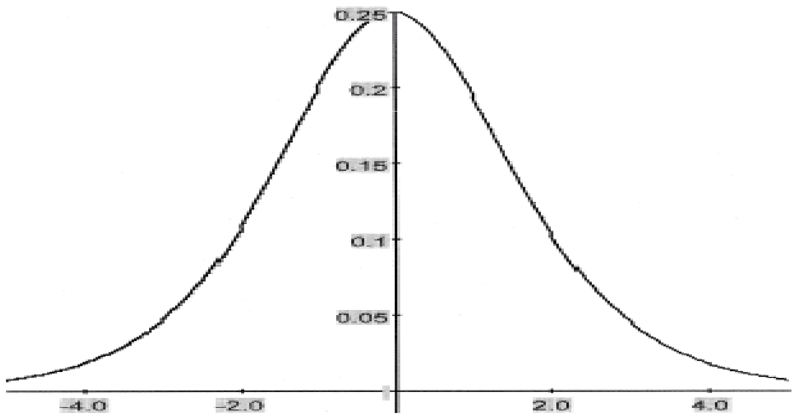


Introduction to the Probit model – PDF Probit vs. Logit

- PDF of Probit:



PDF of Logit:



Introduction to the Probit model – The ML principle

- Joint density:

$$f(y \mid x, \beta) = \prod_i F(x_i^{'}\beta)^{y_i} \left[1 - F(x_i^{'}\beta)\right]^{(1-y_i)}$$

$$= \prod_i F_i^{y_i} (1 - F_i)^{1-y_i}$$

- Log likelihood function:

$$\ln L = \sum_i y_i \ln F_i + (1 - y_i) \ln(1 - F_i)$$

Introduction to the Probit model – The ML principle

- The principle of ML: Which value of β maximizes the probability of observing the given sample?

$$\frac{\partial \ln L}{\partial \beta} = \sum_i \left[\frac{y_i f_i}{F_i} + \frac{(1-y_i)(-f_i)}{1-F_i} \right] x_i$$

$$= \sum_i \left[\frac{y_i - F_i}{F_i(1-F_i)} f_i \right] x_i$$

$$= 0$$

Introduction to the Probit model – Example

- Example taken from Greene, Econometric Analysis, 5. ed. 2003, ch. 17.3.
- 10 observations of a discrete distribution
- Random sample: 5, 0, 1, 1, 0, 3, 2, 3, 4, 1
- PDF:

$$f(x_i,\theta)=\frac{e^{-\theta}\theta^{x_i}}{x_i!}$$

- Joint density :

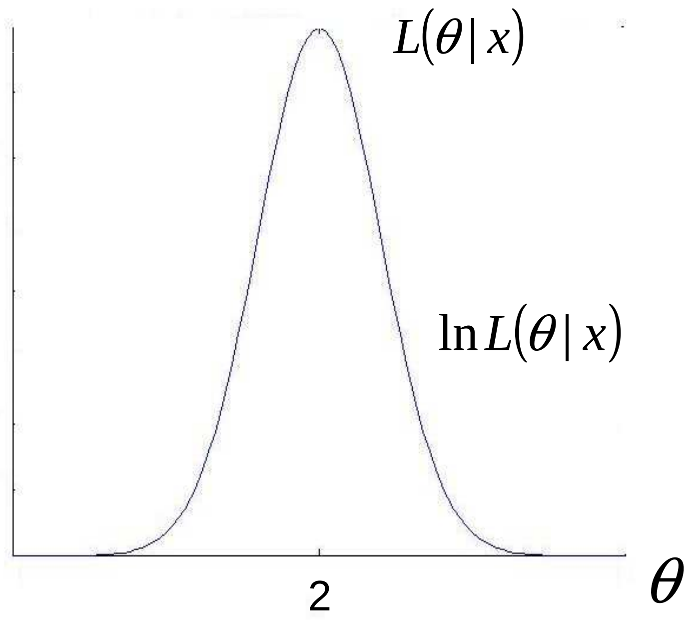
$$f(x_1,x_2,\ldots,x_{10}\,|\,\theta)=\prod_{i=1}^{10} f(x_i,\theta)=\frac{e^{-10\theta}\cdot\theta^{\sum_i x_i}}{\prod_{i=1}^{10} x_i!}=\frac{e^{-10\theta}\cdot\theta^{20}}{207{,}36}$$

- Which value of θ makes occurance of the observed sample most probable?

Introduction to the Probit model – Example

$$\ln L(\theta) = -10\theta + 20\ln\theta - 12{,}242$$

$$\frac{d\ln L(\theta)}{d\theta} = -10 + \frac{20}{\theta} = 0$$



$$\frac{d^2 \ln L(\theta)}{d\theta^2} = -\frac{20}{\theta^2}$$

$$\Rightarrow Maximum$$

Application

- Analysis of the effect of a new teaching method in economic sciences
- Data:

Beobachtung	GPA	TUCE	PSI	Grade	Beobachtung	GPA	TUCE	PSI	Grade
1	2,66	20	0	0	17	2,75	25	0	0
2	2,89	22	0	0	18	2,83	19	0	0
3	3,28	24	0	0	19	3,12	23	1	0
4	2,92	12	0	0	20	3,16	25	1	1
5	4	21	0	1	21	2,06	22	1	0
6	2,86	17	0	0	22	3,62	28	1	1
7	2,76	17	0	0	23	2,89	14	1	0
8	2,87	21	0	0	24	3,51	26	1	0
9	3,03	25	0	0	25	3,54	24	1	1
10	3,92	29	0	1	26	2,83	27	1	1
11	2,63	20	0	0	27	3,39	17	1	1
12	3,32	23	0	0	28	2,67	24	1	0
13	3,57	23	0	0	29	3,65	21	1	1
14	3,26	25	0	1	30	4	23	1	1
15	3,53	26	0	0	31	3,1	21	1	0
16	2,74	19	0	0	32	2,39	19	1	1

Source: Spector, L. and M. Mazzeo, Probit Analysis and Economic Education. In: Journal of Economic Education, 11, 1980, pp.37-44

Application – Variables

- **Grade**
  Dependent variable. Indicates whether a student improved his grades after the new teaching method PSI had been introduced (0 = no, 1 = yes).

- **PSI**
  Indicates if a student attended courses that used the new method (0 = no, 1 = yes).

- **GPA**
  Average grade of the student

- **TUCE**
  Score of an intermediate test which shows previous knowledge of a topic.

Application – Estimation

- Estimation results of the model (output from Stata):

```
. probit  grade psi tuce gpa

Iteration 0:   log likelihood =  -20.59173
Iteration 1:   log likelihood = -13.315851
Iteration 2:   log likelihood = -12.832843
Iteration 3:   log likelihood = -12.818826
Iteration 4:   log likelihood = -12.818803

Probit estimates                                  Number of obs   =         32
                                                  LR chi2(3)      =      15.55
                                                  Prob > chi2     =     0.0014
Log likelihood = -12.818803                       Pseudo R2       =     0.3775

------------------------------------------------------------------------------
       grade |      Coef.   Std. Err.       z     P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
         psi |   1.426332    .595037     2.40   0.017      .2600814    2.592583
        tuce |   .0517289   .0838901     0.62   0.537     -.1126927    .2161506
         gpa |    1.62581   .6938818     2.34   0.019      .2658269    2.985794
       _cons |   -7.45232   2.542467    -2.93   0.003     -12.43546   -2.469177
------------------------------------------------------------------------------
```

Application – Discussion

- ML estimator: Parameters were obtained by maximization of the log likelihood function.
  Here: 5 iterations were necessary to find the maximum of the log likelihood function (-12.818803)

- Interpretation of the estimated coefficients:
  - Estimated coefficients do not quantify the influence of the rhs variables on the probability that the lhs variable takes on the value one.
  - Estimated coefficients are parameters of the latent model.

Coefficients and marginal effects

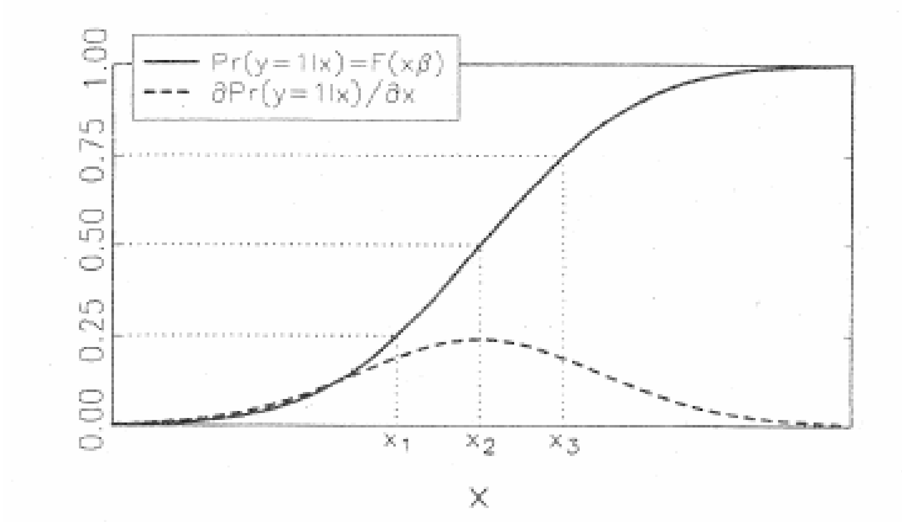
- The marginal effect of a rhs variable is the effect of an unit change of this variable on the probability P(Y = 1|X = x), given that all other rhs variables are constant:

$$\frac{\partial P(y_i = 1 \mid x_i)}{\partial x_i} = \frac{\partial E(y_i \mid x_i)}{\partial x_i} = \varphi(x_i' \beta)\beta$$

- Recap: The slope parameter of the linear regression model measures directly the marginal effect of the rhs variable on the lhs variable.

Coefficients and marginal effects

- The marginal effect depends on the value of the rhs variable.
- Therefore, there exists an individual marginal effect for each person of the sample:



Coefficients and marginal effects – Computation

- Two different types of marginal effects can be calculated:
  - Average marginal effect
    Stata command: `margin`

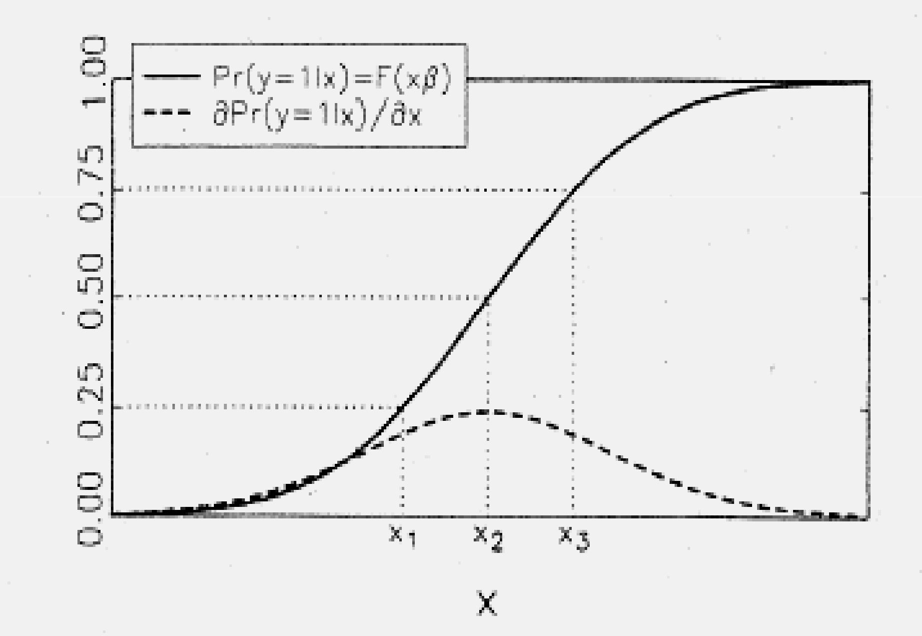
```
Marginal effects on Prob(grade==1) after probit
```

grade	Coef.	Std. Err.	z	P>|z|	[95% Conf.	Interval]
gpa	.3637883	.1129461	3.22	0.001	.1424181	.5851586
tuce	.011476	.0184085	0.62	0.533	-.024604	.047556
psi	.3737518	.1399912	2.67	0.008	.0993741	.6481295

  - Marginal effect at the mean:
    Stata command: `mfx compute`

Coefficients and marginal effects – Computation

- Principle of the computation of the average marginal effects:



- Average of individual marginal effects

Coefficients and marginal effects – Computation

- Computation of average marginal effects depends on type of rhs variable:
  - Continuous variables like TUCE and GPA:

$$AME = \frac{1}{n}\sum_{i=1}^{n}\varphi(x_i^{'}\beta)\beta$$

  - Dummy variable like PSI:

$$AME = \frac{1}{n}\sum_{i=1}^{n}\left[\Phi(x_i^{'}\beta \mid x_i^{k}=1) - \Phi(x_i^{'}\beta \mid x_i^{k}=0)\right]$$

Coefficients and marginal effects – Interpretation

- Interpretation of average marginal effects:
  - Continuous variables like TUCE and GPA:
    An infinitesimal change of TUCE or GPA changes the probability that the lhs variable takes the value one by X%.
  - Dummy variable like PSI:
    A change of PSI from zero to one changes the probability that the lhs variable takes the value one by X percentage points.

Coefficients and marginal effects – Interpretation

Variable	Estimated marginal effect	Interpretation
GPA	0.364	If the average grade of a student goes up by an infinitesimal amount, the probability for the variable grade taking the value one rises by 36.4 %.
TUCE	0.011	Analog to GPA,with an increase of 1.1%.
PSI	0.374	If the dummy variable changes from zero to one, the probability for the variable grade taking the value one rises by 37.4 ppts.

Coefficients and marginal effects – Significance

- Significance of a coefficient: test of the hypothesis whether a parameter is significantly different from zero.
- The decision problem is similar to the t-test, wheras the probit test statistic follows a standard normal distribution. The z-value is equal to the estimated parameter divided by its standard error.
- Stata computes a p-value which shows directly the significance of a parameter:

	z-value	p-value	Interpretation
GPA :	3.22	0.001	***significant***
TUCE:	0,62	0,533	***insignificant***
PSI:	2,67	0,008	***significant***

Coefficients and marginal effects

- Only the average of the marginal effects is displayed.
- The individual marginal effects show large variation:

Descriptive statistics for individual marginal effects

	Mean	SD	Min	Max
gpa	0.36379	0.21358	0.06783	0.64807
tuce	0.01148	0.00687	0.00209	0.02063
psi	0.37375	0.12878	0.06042	0.51959

Stata command: `margin, table`

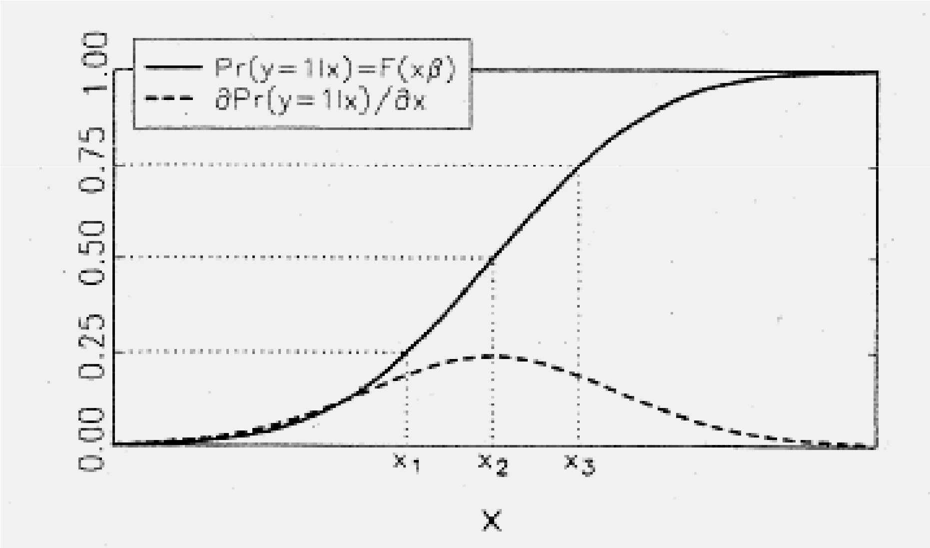
Coefficients and marginal effects

- Variation of marginal effects may be quantified by the confidence intervals of the marginal effects.
- In which range one can expect a coefficient of the population?
- In our example:

	Estimated coefficient	Confidence interval (95%)
GPA:	0,364	- 0,055 - 0,782
TUCE:	0,011	- 0,002 - 0,025
PSI:	0,374	0,121 - 0,626

Coefficients and marginal effects

- What is calculated by `mfx`?
- Estimation of the marginal effect at the sample mean.



Sample mean

Goodness of fit

- Goodness of fit may be judged by McFaddens Pseudo R^2.
- Measure for proximity of the model to the observed data.
- Comparison of the estimated model with a model which only contains a constant as rhs variable.
  - $\ln \hat{L}(M_{Full})$: Likelihood of model of interest.
  - $\ln \hat{L}(M_{Intercept})$: Likelihood with all coefficients except that of the intercept restricted to zero.
  - It always holds that $\ln \hat{L}(M_{Full}) \geq \ln \hat{L}(M_{Intercept})$

Goodness of fit

- The Pseudo R^2 is defined as:

$$Pseudo R^2 = R^2_{McF} = 1 - \frac{\ln \hat{L}(M_{Full})}{\ln \hat{L}(M_{Intercept})}$$

- Similar to the R^2 of the linear regression model, it holds that $0 \leq R^2_{McF} \leq 1$

- An increasing Pseudo R^2 may indicate a better fit of the model, whereas no simple interpretation like for the R^2 of the linear regression model is possible.

Goodness of fit

- A high value of R^2_{McF} does not necessarily indicate a good fit, however, as $R^2_{McF} = 1$ if $\ln \hat{L}(M_{Full}) = 0$.
- R^2_{McF} increases with additional rhs variables. Therefore, an adjusted measure may be appropriate:

$$PseudoR^2_{adjusted} = \overline{R}^2_{McF} = 1 - \frac{\ln \hat{L}(M_{Full}) - K}{\ln \hat{L}(M_{Intercept})}$$

- Further goodness of fit measures: R^2 of McKelvey and Zavoinas, Akaike Information Criterion (AIC), etc. See also the Stata command `fitstat`.

Hypothesis tests

- Likelihood ratio test: possibility for hypothesis testing, for example for variable relevance.
- Basic principle: Comparison of the log likelihood functions of the unrestricted model ($\ln L_U$) and that of the restricted model ($\ln L_R$)
- Test statistic: $LR = -2\ln\lambda = -2(\ln L_R - \ln L_U) \sim \chi^2(K)$

$$\lambda = \frac{L_R}{L_U} \qquad 0 \leq \lambda \leq 1$$

- The test statistic follows a χ^2 distribution with degrees of freedom equal to the number of restrictions.

Hypothesis tests

- Null hypothesis: All coefficients except that of the intercept are equal to zero.
- In the example: LR $\chi^2(3) = 15,55$
- Prob > chi2 = 0.0014
- Interpretation: The hypothesis that all coefficients are equal to zero can be rejected at the 1 percent significance level.